Indian Statistical Institute, Bangalore
M. Math.I Year, First Semester
Mid-Sem Examination
Analysis of Several Variables

Time: 3 hours September 11, 2009 Instructor: B.Bagchi

Maximum Marks 100

1. (a) Define the Cantor set $C \subseteq [0,1]$, and show that $C \times C$ is homeomorphic to C. Conclude that C^k is homeomorphic to C for all $k \geq 2$.

   (b) Show that there is a continuous map from C onto $[0,1]$.

   (c) Show that every continuous map from C into $[0,1]$ extends to a continuous map from $[0,1]$ into itself.

   (d) Use the previous parts to show that there is a continuous map from $[0,1]$ onto $[0,1]^k$. [8+4+4+4=20]

2. Let $\mathbb{X} \subseteq \mathbb{R}^d$. Show that every open cover of X has a countable subcover. [20]

3. (a) Define the total derivative of $f : \Omega \longrightarrow \mathbb{R}^e$($\Omega$ an open subset of $\mathbb{R}^d$) at a point $x \varepsilon \Omega$. Show that, if the total derivative $f'(x)$ exists, then all the d partial derivatives of f exist at x, and find an expression for $f'(x)$ in terms of these partial derivatives.

   (b) If $f_1, f_2, \cdots f_d$ are d differentiable functions $\mathbb{R} \longrightarrow \mathbb{R}$ then show that the function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $f(x) = f_1(x_1) + \cdots + f_d(x_d)$ is differentiable everywhere, and the derivative is given by $f'(x)v = \sum\limits_{i=1}^{d} v_i f_i'(x_i), v \epsilon \mathbb{R}^d$. [10+10=20]

4. (a) Let $\Omega \subseteq \mathbb{R}^d$ be open, $f : \mathbb{R}^d \longrightarrow \mathbb{R}^e$. If, for some $i, j (1 \leq i \neq j \leq d^1)$ the partial derivative $\partial_i \partial j f$ and $\partial_j \partial i f$ exist and are continuous on Ω then show that they are equal.

   (b) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Then show that $\partial_1\, \partial_2\, f$ and $\partial_2\, \partial_1\, f$ exist throughout $\mathbb{R}^2$, but they are not equal everywhere. [10+10=20]

5. Let Ω be a convex open subset of $\mathbb{R}^d$ and $f : \Omega \longrightarrow \mathbb{R}^e$ be differentiable everywhere. Suppose all the partial derivatives of f are bounded on Ω.

   (a) Show that there is a constant m such that $\|f'(x)\| \leq m$ for all $x \in \Omega$.

   (b) Conclude that there is a constant c such that $\|f(x) - f(y)\| \leq c\|x - y\|$ for all $x, y \in \Omega$. [10+10=20]